

Definition

An exponential generating function for the integer sequence $a(0), a(1), a(2), \dots$ is a function $G(x)$ such that:

$$G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} = a_0 + a_1 \frac{x}{1!} + a_2 \frac{x^2}{2!} + \dots$$

For example:

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots = \exp(x) \Rightarrow \{a_n\}_{n=0}^{\infty} = 1, 1, 1, 1, 1, 1, \dots$$

Properties:

$$A(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$B(x) = \sum_{n=0}^{\infty} b_n \frac{x^n}{n!}$$

$$A(x)B(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \Rightarrow c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}$$

$$A(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$\hat{A}(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$$

$$A(x) = \int_0^{\infty} \exp(-t) \hat{A}(tx) dt$$

Exponential generating functions of some integer sequences

Bell numbers

OEIS [A000110](#)

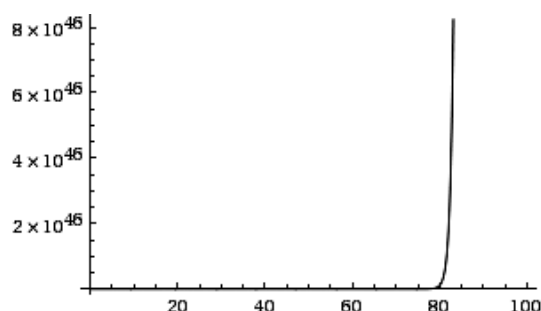
$$\exp(\exp(x - 1))$$

1, 1, 2, 5, 15, 52, 203, 877, 4140, 21147, 115975, 678570, 4213597, 27644437, 190899322, 1382958545, 10480142147, 82864869804, 682076806159, 5832742205057, 51724158235372, 474869816156751, 4506715738447323, 44152005855084346, 445958869294805289, 4638590332229999353, 49631246523618756274

Catalan numbers

OEIS [A000108](#)

$$(I_0(2x) - I_1(2x)) \exp(2x)$$

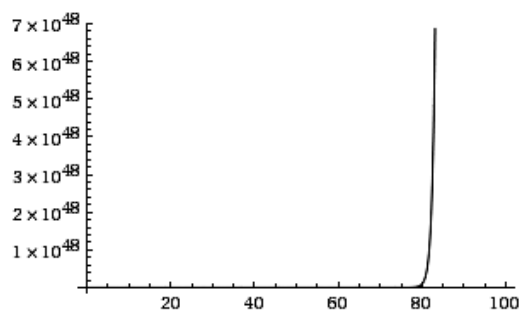


1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900, 2674440, 9694845, 35357670, 129644790, 477638700, 1767263190, 6564120420, 24466267020, 91482563640, 343059613650, 1289904147324, 4861946401452, 18367353072152, 69533550916004, 263747951750360, 1002242216651368

Central binomial coefficients

OEIS [A000984](#)

$$I_0(2x) \exp(2x)$$

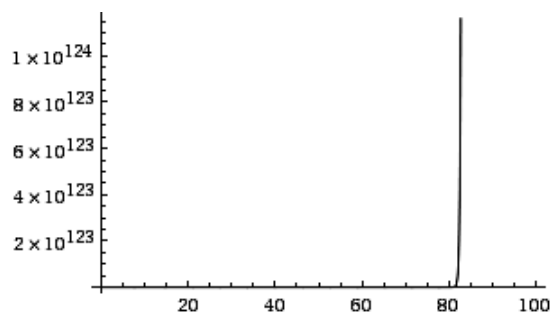


1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, 2704156, 10400600, 40116600, 155117520, 601080390, 2333606220, 9075135300, 35345263800, 137846528820, 538257874440, 2104098963720, 8233430727600, 32247603683100, 126410606437752, 495918532948104, 1946939425648112

Factorial numbers

OEIS [A000142](#)

$$\frac{1}{1-x}$$

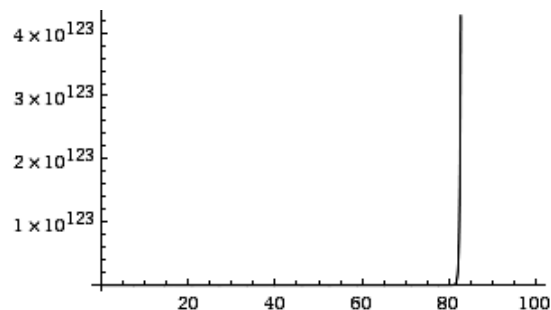


1, 1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, 39916800, 479001600, 6227020800, 87178291200, 1307674368000, 20922789888000, 355687428096000, 6402373705728000, 121645100408832000, 2432902008176640000, 51090942171709440000

Subfactorial

OEIS [A000166](#)

$$\frac{\exp(-x)}{1-x}$$

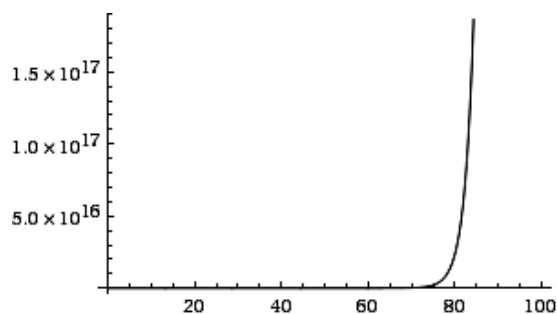


1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, 32071101049, 481066515734, 7697064251745, 130850092279664, 2355301661033953, 44750731559645106, 895014631192902121, 18795307255050944540, 413496759611120779881, 9510425471055777937262

Fibonacci numbers

OEIS [A000045](#)

$$\frac{2 \exp\left(\frac{x}{2}\right) \sinh\left(\frac{\sqrt{5}x}{2}\right)}{\sqrt{5}}$$



0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524578, 5702887, 9227465, 14930352, 24157817, 39088169

Bisection of Fibonacci sequence

OEIS [A001906](#)

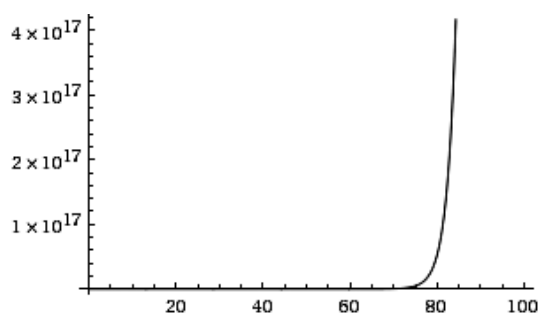
$$\frac{2 \exp\left(\frac{3x}{2}\right) \sinh\left(\frac{\sqrt{5}x}{2}\right)}{\sqrt{5}}$$

0, 1, 3, 8, 21, 55, 144, 377, 987, 2584, 6765, 17711, 46368, 121393, 317811, 832040, 2178309, 5702887, 14930352, 39088169, 102334155, 267914296, 701408733, 1836311903, 4807526976, 12586269025, 32951280099, 86267571272, 225851433717, 591286729879, 1548008755920

Lucas numbers

OEIS [A000032](#)

$$2 \exp\left(\frac{x}{2}\right) \sinh\left(\frac{\sqrt{5}x}{2}\right)$$

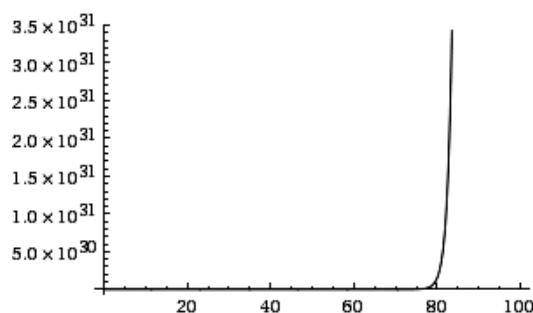


2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, 3571, 5778, 9349, 15127, 24476, 39603, 64079, 103682, 167761, 271443, 439204, 710647, 1149851, 1860498, 3010349, 4870847, 7881196, 12752043, 20633239, 33385282

Pell numbers

OEIS [A000129](#)

$$\frac{\exp(x) \sinh(\sqrt{2}x)}{\sqrt{2}}$$

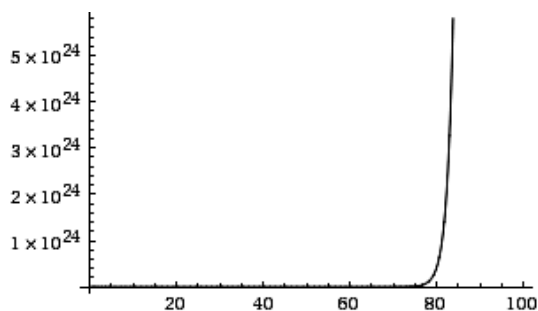


0, 1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, 80782, 195025, 470832, 1136689, 2744210, 6625109, 15994428, 38613965, 93222358, 225058681, 543339720, 1311738121, 3166815962, 7645370045, 18457556052, 44560482149, 107578520350, 259717522849

Jacobsthal numbers

OEIS [A001045](#)

$$\frac{\exp(2x) - \exp(-x)}{3}$$



0, 1, 1, 3, 5, 11, 21, 43, 85, 171, 341, 683, 1365, 2731, 5461, 10923, 21845, 43691, 87381, 174763, 349525, 699051, 1398101, 2796203, 5592405, 11184811, 22369621, 44739243, 89478485, 178956971, 357913941, 715827883, 1431655765, 2863311531, 5726623061

Numerators of continued fraction convergents to $\sqrt{2}$

OEIS [A001333](#)

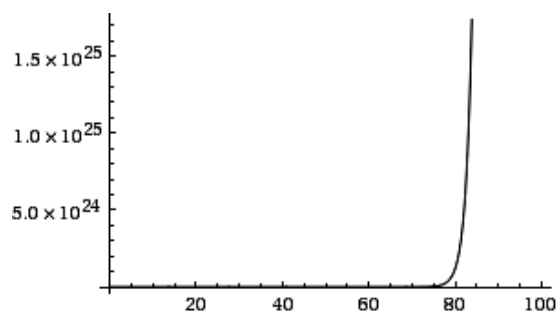
$$\exp(x) \cosh(x\sqrt{2})$$

1, 1, 3, 7, 17, 41, 99, 239, 577, 1393, 3363, 8119, 19601, 47321, 114243, 275807, 665857, 1607521, 3880899, 9369319, 22619537, 54608393, 131836323, 318281039, 768398401, 1855077841, 4478554083, 10812186007, 26102926097, 63018038201, 152139002499

Powers of 2

OEIS [A000079](#)

$$\exp(2x)$$

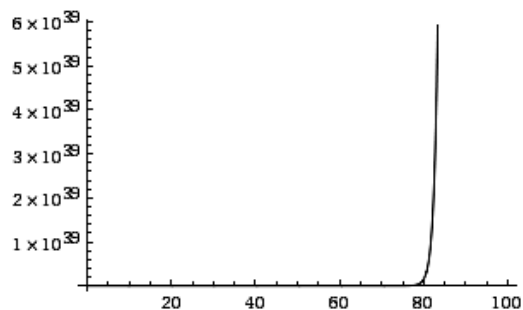


1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192, 16384, 32768, 65536, 131072, 262144, 524288, 1048576, 2097152, 4194304, 8388608, 16777216, 33554432, 67108864, 134217728, 268435456, 536870912, 1073741824, 2147483648, 4294967296, 8589934592

Powers of 3

OEIS [A000244](#)

$$\exp(3x)$$

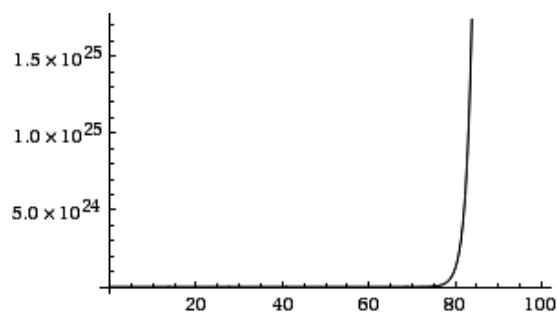


1, 3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, 177147, 531441, 1594323, 4782969, 14348907, 43046721, 129140163, 387420489, 1162261467, 3486784401, 10460353203, 31381059609, 94143178827, 282429536481, 847288609443, 2541865828329, 7625597484987

$$\underline{2^n - 1}$$

OEIS [A000225](#)

$$(\exp(x) - 1) \exp(x)$$



0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, 16383, 32767, 65535, 131071, 262143, 524287, 1048575, 2097151, 4194303, 8388607, 16777215, 33554431, 67108863, 134217727, 268435455, 536870911, 1073741823, 2147483647, 4294967295

O, 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111, 111111111, 1111111111, 11111111111,
111111111111, 1111111111111, 11111111111111, 111111111111111, 1111111111111111,
11111111111111111, 111111111111111111, 1111111111111111111, 11111111111111111111,
11111111111111111111

The even numbers

OEIS [A005843](#)

$$2x \exp(x)$$

0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 76, 78, 80, 82, 84, 86, 88, 90, 92, 94, 96, 98, 100, 102, 104, 106, 108, 110, 112, 114, 116, 118, 120

Floor(n/2)

OEIS [A004526](#)

$$\frac{(2x - 1) \exp(x) + \exp(-x)}{4}$$

0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 10, 11, 11, 12, 12, 13, 13, 14, 14, 15, 15, 16, 16, 17, 17, 18, 18, 19, 19, 20, 20, 21, 21, 22, 22, 23, 23, 24, 24, 25, 25, 26, 26, 27, 27, 28, 28, 29, 29, 30, 30, 31, 31, 32, 32, 33, 33, 34, 34, 35, 35, 36, 36

Pronic numbers

OEIS [A002378](#)

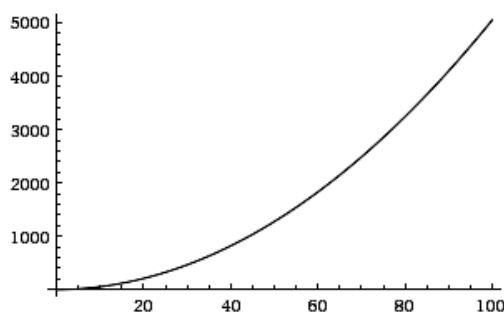
$$x(2 + x) \exp(x)$$

0, 2, 6, 12, 20, 30, 42, 56, 72, 90, 110, 132, 156, 182, 210, 240, 272, 306, 342, 380, 420, 462, 506, 552, 600, 650, 702, 756, 812, 870, 930, 992, 1056, 1122, 1190, 1260, 1332, 1406, 1482, 1560, 1640, 1722, 1806, 1892, 1980, 2070, 2162, 2256, 2352, 2450, 2550

Triangular numbers

OEIS [A000217](#)

$$\frac{x(2+x)\exp(x)}{2}$$

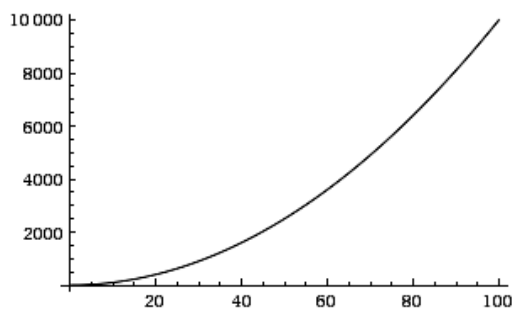


0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136, 153, 171, 190, 210, 231, 253, 276, 300, 325, 351, 378, 406, 435, 465, 496, 528, 561, 595, 630, 666, 703, 741, 780, 820, 861, 903, 946, 990, 1035, 1081, 1128, 1176, 1225, 1275, 1326, 1378, 1431

The squares

OEIS [A000290](#)

$$x(1+x)\exp(x)$$



0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, 961, 1024, 1089, 1156, 1225, 1296, 1369, 1444, 1521, 1600, 1681, 1764, 1849, 1936, 2025, 2116, 2209, 2304, 2401, 2500

Pentagonal numbers

OEIS [A000326](#)

$$\frac{x(2 + 3x) \exp(x)}{2}$$

0, 1, 5, 12, 22, 35, 51, 70, 92, 117, 145, 176, 210, 247, 287, 330, 376, 425, 477, 532, 590, 651, 715, 782, 852, 925, 1001, 1080, 1162, 1247, 1335, 1426, 1520, 1617, 1717, 1820, 1926, 2035, 2147, 2262, 2380, 2501, 2625, 2752, 2882, 3015, 3151

Hexagonal numbers

OEIS [A000384](#)

$$x(1 + 2x) \exp(x)$$

0, 1, 6, 15, 28, 45, 66, 91, 120, 153, 190, 231, 276, 325, 378, 435, 496, 561, 630, 703, 780, 861, 946, 1035, 1128, 1225, 1326, 1431, 1540, 1653, 1770, 1891, 2016, 2145, 2278, 2415, 2556, 2701, 2850, 3003, 3160, 3321, 3486, 3655, 3828, 4005, 4186, 4371, 4560

Heptagonal numbers

OEIS [A000566](#)

$$\frac{x(2 + 5x) \exp(x)}{2}$$

0, 1, 7, 18, 34, 55, 81, 112, 148, 189, 235, 286, 342, 403, 469, 540, 616, 697, 783, 874, 970, 1071, 1177, 1288, 1404, 1525, 1651, 1782, 1918, 2059, 2205, 2356, 2512, 2673, 2839, 3010, 3186, 3367, 3553, 3744, 3940, 4141, 4347, 4558, 4774, 4995, 5221, 5452, 5688

Octagonal numbers

OEIS [A000567](#)

$$x(1 + 3x) \exp(x)$$

0, 1, 8, 21, 40, 65, 96, 133, 176, 225, 280, 341, 408, 481, 560, 645, 736, 833, 936, 1045, 1160, 1281, 1408, 1541, 1680, 1825, 1976, 2133, 2296, 2465, 2640, 2821, 3008, 3201, 3400, 3605, 3816, 4033, 4256, 4485, 4720, 4961, 5208, 5461

Central polygonal numbers

OEIS [A000124](#)

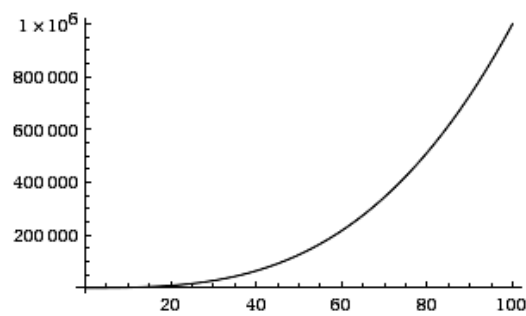
$$\frac{(2 + 2x + x^2) \exp(x)}{2}$$

1, 2, 4, 7, 11, 16, 22, 29, 37, 46, 56, 67, 79, 92, 106, 121, 137, 154, 172, 191, 211, 232, 254, 277, 301, 326, 352, 379, 407, 436, 466, 497, 529, 562, 596, 631, 667, 704, 742, 781, 821, 862, 904, 947, 991, 1036, 1082, 1129, 1177, 1226, 1276, 1327

The cubes

OEIS [A000578](#)

$$x(1 + 3x + x^2) \exp(x)$$

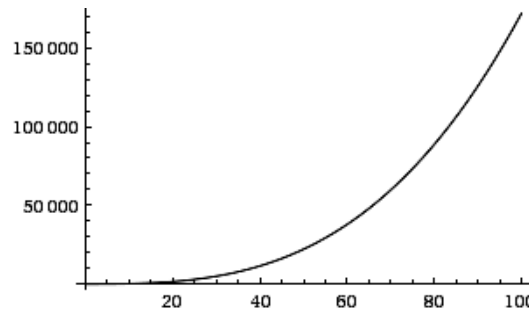


0, 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, 1728, 2197, 2744, 3375, 4096, 4913, 5832, 6859, 8000, 9261, 10648, 12167, 13824, 15625, 17576, 19683, 21952, 24389, 27000, 29791, 32768, 35937, 39304, 42875, 46656, 50653, 54872, 59319, 64000

Tetrahedral numbers

OEIS [A000292](#)

$$\frac{x(6 + 6x + x^2) \exp(x)}{6}$$



0, 1, 4, 10, 20, 35, 56, 84, 120, 165, 220, 286, 364, 455, 560, 680, 816, 969, 1140, 1330, 1540, 1771, 2024, 2300, 2600, 2925, 3276, 3654, 4060, 4495, 4960, 5456, 5984, 6545, 7140, 7770, 8436, 9139, 9880, 10660, 11480, 12341, 13244, 14190, 15180

Octahedral numbers

OEIS [A005900](#)

$$\frac{x(3 + 6x + 2x^2) \exp(x)}{3}$$

0, 1, 6, 19, 44, 85, 146, 231, 344, 489, 670, 891, 1156, 1469, 1834, 2255, 2736, 3281, 3894, 4579, 5340, 6181, 7106, 8119, 9224, 10425, 11726, 13131, 14644, 16269, 18010, 19871, 21856, 23969, 26214, 28595, 31116, 33781, 36594, 39559, 42680

Icosahedral numbers

OEIS [A006564](#)

$$\frac{x(2 + 10x + 5x^2) \exp(x)}{2}$$

1, 12, 48, 124, 255, 456, 742, 1128, 1629, 2260, 3036, 3972, 5083, 6384, 7890, 9616, 11577, 13788, 16264, 19020, 22071, 25432, 29118, 33144, 37525, 42276, 47412, 52948, 58899, 65280, 72106, 79392, 87153, 95404, 104160, 113436, 123247, 133608

Dodecahedral numbers

OEIS [A006566](#)

$$\frac{x(2 + 18x + 9x^2) \exp(x)}{2}$$

0, 1, 20, 84, 220, 455, 816, 1330, 2024, 2925, 4060, 5456, 7140, 9139, 11480, 14190, 17296, 20825, 24804, 29260, 34220, 39711, 45760, 52394, 59640, 67525, 76076, 85320, 95284, 105995, 117480, 129766, 142880, 156849, 171700, 187460, 204156

Square pyramidal numbers

OEIS [A000330](#)

$$\frac{x(6 + 9x + 2x^2) \exp(x)}{6}$$

0, 1, 5, 14, 30, 55, 91, 140, 204, 285, 385, 506, 650, 819, 1015, 1240, 1496, 1785, 2109, 2470, 2870, 3311, 3795, 4324, 4900, 5525, 6201, 6930, 7714, 8555, 9455, 10416, 11440, 12529, 13685, 14910, 16206, 17575, 19019, 20540, 22140, 23821, 25585, 27434, 29370

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